EUCLIDEAN GEOMETRIC INVARIANT OF FRAMED KNOTS IN MANIFOLDS

JÉRÔME DUBOIS, IGOR G. KOREPANOV AND EVGENIY V. MARTYUSHEV

Abstract. We present an invariant of a three–dimensional manifold with a framed knot in it based on the Reidemeister torsion of an acyclic complex of Euclidean geometric origin. To show its nontriviality, we calculate the invariant for some framed (un)knots in lens spaces. An important feature of our work is that we are not using any nontrivial representation of the manifold fundamental group or knot group.

1. Introduction

Reidemeister torsion made its first appearance in 1935, in the work of Reidemeister [Rei35] on the combinatorial classification of the three–dimensional lens spaces by means of the based simplicial chain complex of the universal cover. A radically different approach was proposed by one of the authors of the present paper who discovered in paper [Kor01] an invariant of three–dimensional manifolds, based on introducing Euclidean geometry into the simplices of a manifold triangulation. One obvious mathematical ingredient in the constructions of [Kor01] was an algebraic formula for infinitesimal geometric quantities corresponding naturally to a Pachner move $2 \to 3$ (see Figures 3 and 4 and Formula (12) below). Gradually, it became clear [Kor02, Kor03] that there was really one more basic ingredient — the theory of Reidemeister torsion — behind the construction in [Kor01].

The key role in this construction, which was initially proposed for *closed* manifolds, was played by matrix $(\partial\omega_a/\partial l_b)$ of partial derivatives of the so–called deficit angles ω_a with respect to the edge lengths l_b, where subscripts a and b parametrized the edges (see Section 3 for detailed definitions). The next natural direction of research should be the investigation of these invariants for manifolds *with boundary*. In choosing this direction, one is guided by the idea of constructing eventually a topological field theory according

1991 *Mathematics Subject Classification.* 57M27; 57Q99.

Key words and phrases. Pachner moves; Reidemeister torsion; Framed knots; Triangulation and pseudotriangulation; Differential relations for geometric values.

to Atiyah's axioms [Ati88] (or some modification of them) where, as is well known, the boundary of a manifold plays an important role.

Here we are trying to make a first step in this direction. To be more precise, we investigate *relative* invariants corresponding to a pair consisting of a closed triangulated manifold and a framed knot in it. The relation to manifolds with boundary is as follows. We take a special (pseudo)triangulation of a closed oriented three–dimensional manifold containing two tetrahedra which form a chain (see below and in particular Figures 1 and 2) and whose special edges can be viewed as a *framed* knot (see Section 2 for a detailed explanation). The manifold with boundary is the initial manifold minus these two tetrahedra.

Our invariants appear from the same matrix $(\partial\omega_a/\partial l_b)$ as before ([Kor01]) but with some additional structure. One can observe that both rows and columns of this matrix correspond to edges of the triangulation. We select some "distinguished" edges and then treat in a special way both the rows and columns corresponding to them. According to Atiyah's axioms, these distinguished edges are chosen so that they lie in the boundary of the manifold. This boundary is a triangulated torus, so we can assume that we are considering a manifold with a toric boundary where the triangulation specifies the meridian and the parallel (or "framing", see Section 2 for details).

To put our work in context, we briefly recall some methods and results from our earlier papers. The invariant considered in [Kor01] makes use of the largest nonvanishing minor of matrix $(\partial\omega_a/\partial l_b)$; some special construction was used to eliminate the non-uniqueness in the choice of this minor, and it has been shown later in [Kor02, Section 2] and [Kor03, Section 2] that this construction consisted, essentially, in taking the *torsion of an acyclic complex* built from differentials of geometric quantities. It is interesting to mention that the main objective of papers [Kor02, Kor03] was to generalize ideas from [Kor01] to four–dimensional manifolds. The algebra necessary for four-manifolds was, naturally, more complicated. However, a careful study of this complicated situation has lead to important clarifications not only for the four–dimensional, but also for the three–dimensional case.

There exists also a version of this invariant using a universal cover of the manifold and nontrivial representations of the fundamental group $\pi_1(M)$ into the group of motions of three–dimensional Euclidean space [KM02]. In this way, an invariant which seems to be related to Reidemeister torsion has been obtained. A good illustration is the following formula for the invariant of lens spaces proved recently in [Mar]:

$$\operatorname{Inv}_k\left(L(p,q)\right) = -\frac{1}{p^2}\left(4\sin\frac{\pi k}{p}\sin\frac{\pi kq}{p}\right)^4. \tag{1}$$

Here $L(p,q)$ is a three–dimensional lens space; the subscript k takes integer values from 1 to the integral part of $p/2$; the invariant consists of real numbers corresponding to each of these k. One can check that Formula (1) is essentially minus the square of the Reidemeister torsion of $L(p,q)$ in the adjoint representation associated to the representation ρ_k which brings the generator (see the first paragraph of Subsection 6.2 for its definition) of the fundamental group of $L(p,q)$ to $e^{2\pi ik/p} \in \mathrm{U}(1)$.

The invariants appearing from nontrivial representations of $\pi_1(M)$ form an important area of research. This applies to "usual" Reidemeister torsion for manifolds and knots [Dub05] as well as "geometric" torsion. One can find some conjectures, concerning the relation of "geometric" and "usual" invariants constructed using Reidemeister torsions and based on computer calculations, in paper [Mar04]. Note however that the important feature of the present paper is that we are not using any nontrivial representation of the manifold fundamental group or knot group. Formula (1) has been cited here only to illustrate the fact that, in some situations, the invariant obtained from "geometric" torsion can be expressed through the "usual" Reidemeister torsion.

As for the present paper, its direct aim is to introduce, in the outlined way, an invariant of a pair consisting in a manifold and a framed knot in it, and show its nontriviality on some simplest examples of "unknots", i.e. simplest closed contours, in lens spaces. From a more global standpoint, the aim of the paper is to investigate the possibility of building a meaningful topological field theory on the basis of differential relations between geometric values put in correspondence to the elements of a manifold triangulation, and stimulate further research (see Section 10).

Organization. As we attach some special role to some tetrahedra in the triangulation, we do not want to touch them when transforming a manifold triangulation into another one using Pachner moves. So, we need to prove that Pachner moves *not touching those tetrahedra at any step* are enough to come to any new triangulation. These moves are called *relative Pachner moves* in this paper. This is the technical part of the paper, which is done in Section 2. In Section 3, we define geometric values needed for the construction of an acyclic complex, and in Section 4 we show how to construct this complex and prove the invariance of its torsion, multiplied by some geometric values, with respect to relative Pachner moves. In Section 5, we show how to change the knot framing within our construction, and how this affects the acyclic complex. The next sections consist in our examples: framed "unknots" in lens spaces. In Section 6, we define some standard triangulations of lens spaces and show how some special framed unknots appear readily within such a triangulation. In Section 7, we explain the general structure of matrix $(\partial\omega_a/\partial l_b)$ for a lens space and then in Section 8 we calculate the invariants for the mentioned unknots in a lens space $L(p,q)$ with a "simplest" framing, while in Section 9 we do the same for all framings. In Section 10, we discuss the results of our paper.

2. Pseudotriangulation for a manifold with a framed knot in it and relative Pachner moves

We consider a closed oriented three–manifold M and a triangulation of it containing a distinguished chain of two tetrahedra $ABCD$ of one of the forms depicted in Figures 1 and 2.

These two tetrahedra can either have the same orientation, as in Figure 1, or the opposite orientations, as in Figure 2.

Strictly speaking, what we are considering is not a triangulation in the sense of Lickorish's paper [Lic99] but a *pseudo*triangulation. As we plentifully use the results of [Lic99], from now on we adopt this stricter language. We will construct invariants of such pseudotriangulations with respect to certain *Pachner moves* (see Subsection 2.1). Our construction of the invariant require to adopt the following convention (see Subsection 3.1 for details).

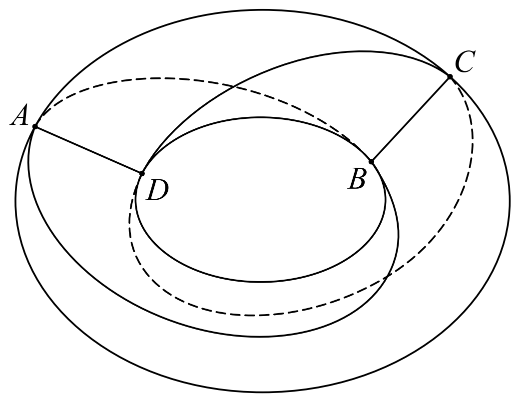

Figure 1. A chain of two identically oriented tetrahedra $ABCD$

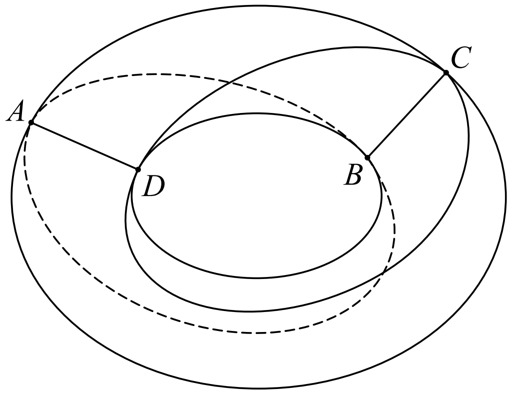

FIGURE 2. A chain of two oppositely oriented tetrahedra $ABCD$

Convention. Any pseudotriangulation considered in this paper, including those which appear below at any step of a sequence of Pachner moves, is required to possess the following property: *all* vertices of any tetrahedron are different.

Remark 1. Observe that proper triangulations automatically satisfy the preceding convention. So, the first important thing to notice is that the *initial* pseudotriangulation we use must obey the requirement of the preceding convention. The second important place where we will have to take care about this convention is Subsection 2.2.1.

2.1. **Relative Pachner moves.** To select a special chain of two tetrahedra as depicted in Figures 1 and 2 essentially means the same as to select a *framed knot* in M. To be exact, there is a knot with *two* framings given either by two closed lines (which we imagine as close to each other) ACA and DBD, or by the two lines ABA and DCD. In the case of the same orientation of the two tetrahedra, these possibilities lead to framings which differ in one full revolution (of the ribbon between two lines), so we choose the "intermediate" framing differing from them both in one–half of a revolution as the framing corresponding to our picture. In the case of the opposite orientations of the two tetrahedra, both ways simply give the same framing.

Our aim is to construct an invariant of a pair (M, K), where K is a framed knot in M, starting from a pseudotriangulation of M containing two distinguished tetrahedra as in Figures 1 and 2. To achieve this, we will construct in Section 4 a value not changing under Pachner moves on pseudotriangulation of M *not touching the distinguished tetrahedra* of Figures 1 and 2. By "not touching" we understand those moves that do not replace either of the two tetrahedra in Figures 1 and 2 with any other tetrahedra, and we call such moves *relative Pachner moves*.

Recall that Pachner moves are elementary rebuildings of a *closed* triangulated manifold. There are four such moves on three–dimensional manifolds. Two of them are illustrated in Figures 3 and 4,

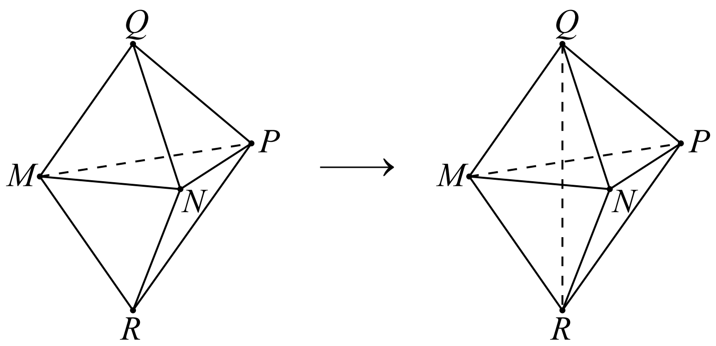

FIGURE 3. A $2 \to 3$ Pachner move in three dimensions

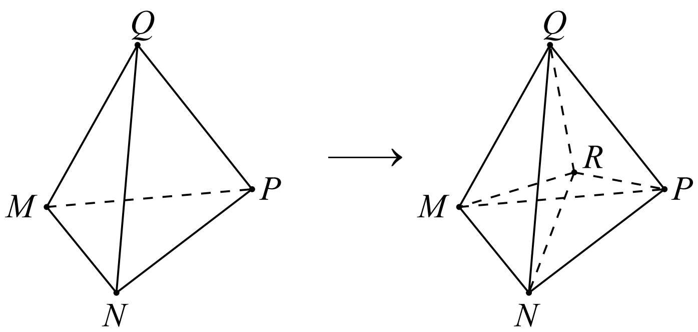

FIGURE 4. A $1 \to 4$ Pachner move in three dimensions

and the other two are inverse to these. The move in Figure 3 replaces the two adjacent tetrahedra $MNPQ$ and $RMNP$ with three new tetrahedra: $MNRQ$, $NPRQ$, and $PMRQ$. The move in Figure 4 replaces one tetrahedron $MNPQ$ with four of them: $MNPR$, $MNRQ$, $MPQR$, and $NPRQ$.

Elementary transformations for triangulated manifolds *with boundary* are known as *shellings* and *inverse shellings*, see [Lic99]. Although we do mention some of such transformations in Section 5, we leave the development of the techniques needed for work with shellings in our context to further papers.

So, the main objective of the present section is to prove the following technical result.

Theorem 1. *Two pseudotriangulations with the same chain of two distinguished tetrahedra $ABCD$, as depicted in Figures 1 and 2, are related by a sequence of relative Pachner moves.*

2.2. **Proof of Theorem 1.** The proof of Theorem 1 requires some technical facts described in the following paragraph.

2.2.1. *From pseudotriangulations to triangulations.* To prove Theorem 1 we apply techniques from Lichorish's paper [Lic99]. But to use it we must manipulate triangulated manifolds in the proper sense.

In this subsection we explain a method to pass from a pseudotriangulation to a triangulation in the proper sense, i.e., subdivide the pseudotriangulation in such way that every simplex is unambiguously determined by the set of its vertices, and the boundary of any simplex does not contain any simplex of smaller dimension more than once. Our method will be consistent with the convention about pseudotriangulations adopted in this section. Together with Pachner moves (see Figures 3 and 4), we will use *stellar moves*, see [Lic99, Section 3]. In three dimensions, there is no problem to express the latter in terms of the former (and vice versa).

In order to not touch the two distinguished tetrahedra $ABCD$, we will temporarily remove them from the simplicial complex, together with some neighboring tetrahedra, in the following way.

We can assume that our pseudotriangulation already does not contain any more edges or two–dimensional faces whose all vertices lie in the set $\{A, B, C, D\}$ except those depicted in Figures 1 and 2 — in other case, we can always make obvious stellar subdivisions to ensure this.

Now, starting from a pseudotriangulation containing a chain of two tetrahedra as in Figures 1 and 2, we first do Pachner moves $1 \to 4$ (or, which is the same, stellar subdivisions) in all tetrahedra adjacent to those two in Figures 1 and 2. Thus, there have appeared eight new vertices, we call them $N_1, \dots, N_8$.

Next, we look at the edges in Figures 1 and 2. We are going to make some moves so that the link of each of them contain exactly one vertex between the two N_i. If there were more such vertices, we can eliminate them from the link by doing suitable $2 \to 3$ Pachner moves. Namely, the $2 \to 3$ Pachner move provides a new edge joining N_i directly with a "farther" vertex in the link, thus eliminating from the link the next to N_i vertex, see Figure 5. A

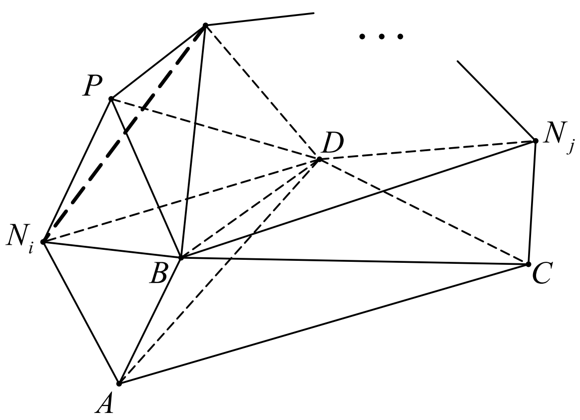

FIGURE 5. The edge drawn in boldface dashed line appears as a result of move $2 \to 3$ and eliminates vertex P from the link of BD

special case is two edges AD and BC: they require such procedure to be applied twice, "on two sides".

This done, we make stellar subdivisions in the two–dimensional faces which are star-products of the edges in Figures 1 and 2 and the vertices lying between the N_i's — two such vertices for each of AD and BC, and one for each of the remaining vertices.

After that, we remove from the resulting simplicial complex those tetrahedra that have *at least two* vertices in the set $\{A, B, C, D\}$. Technically, we can put it as follows. Take these tetrahedra together with all their faces. We let L denote this simplicial subcomplex. Then we remove the subcomplex L from our simplicial complex and take the closure of what remains; let V denote this closure.

Note that the triangulated boundary of V (which is also the boundary of L) can be described as follows: first, *double* the edges AD and BC in Figures 1 and 2 in such way as to make a torus out of the boundary of the tetrahedron chain, and then make a barycentric subdivision of this triangulated torus.

Finally, we subdivide our simplicial complex V, doing, e.g. suitable stellar moves in its simplices but leaving the boundary untouched, so that it becomes a triangulation in the proper sense, as it is required in order to apply the techniques from [Lic99] (it is obvious but important that we can do so). Let W denote the resulting simplicial complex.

2.2.2. *Proof of Theorem 1.* We are now ready to prove Theorem 1.

Proof of Theorem 1. Apply the above procedure to the two initial pseudotriangulations of the PL-manifold M with the same distingushed tetrahedron chain of the form given by Figures 1 and 2. Let W_1 and W_2 be the obtained simplicial complexes.

Obviously, W_1 and W_2 are PL-homeomorphic. Then, according to [Lic99, Theorem 4.5], these simplicial complexes are stellar equivalent. To be more precise, W_2 can be obtained from W_1 by a sequence of stellar moves performed on its simplices.

If we apply the same sequence of stellar moves to the union $W_1 \cup L$, then we clearly obtain $W_2 \cup L$. The subcomplex in Figures 1 and 2 will not be touched by any move in the sequence. Neither is it touched by all the other moves mentioned in this section. So, what remains is to replace the stellar moves with suitable (sequences of) Pachner moves. □

3. Geometric values needed for the acyclic complex

We are now going to construct an acyclic complex which produces the invariant of a three–manifold with a framed knot in it given by a chain of two tetrahedra as in Figures 1 and 2. The complex will be like those in [Kor02, Section 2] and [Kor03, Section 2], but in fact a bit simpler. To be exact, we will construct an algebraic complex which is acyclic in many interesting cases, see a conjecture below in Section 4 (and we hope to be able to modify this algebraic complex into an acyclic one in other cases to be considered in further papers).

Convention. Recall that we are considering an *orientable* manifold M. From now on, we fix a *consistent orientation* for all tetrahedra in the triangulation. The orientation of a tetrahedron is understood here as an ordering of its vertices up to an even permutation; for instance, two tetrahedra $ABCD$ and $EABC$, having a common face ABC, are consistently oriented.

3.1. **Oriented volumes and deficit angles.**

We need the so–called *deficit angles* corresponding to the edges of triangulation. The rest of this section is devoted to explaining these deficit angles and related notions, while the acyclic complex itself will be presented in Section 4.

Recall that we assume that *all* the vertices of any tetrahedron in the pseudotriangulation are different (convention in the beginning of Section 2). Put all the vertices of the pseudotriangulation in $\mathbb{R}^3$ (i.e., we ascribe to each of them three real coordinates) in arbitrary way with only one condition: no four vertices must lie in the same plane. This condition ensures that the geometric quantities we will need — edge lengths and tetrahedron volumes — never vanish.

When we put an oriented tetrahedron $ABCD$ into the Euclidean space $\mathbb{R}^3$ (remember that the vertices A, B, C and D do not lie in the same plane), we can ascribe to it an *oriented volume* denoted V_{ABCD} according to the formula

$$6V_{ABCD} = \overrightarrow{AB} \cdot \overrightarrow{AC} \cdot \overrightarrow{AD} \tag{2}$$

(scalar triple product in the right-hand side). If the sign of the volume defined by Equation (2) of a given tetrahedron is positive, we say that it is put in $\mathbb{R}^3$ *with its orientation preserved*; if it is negative we say that it is put in $\mathbb{R}^3$ *with its orientation changed.*

Now we consider the dihedral angles at the edges of triangulation. We will ascribe a sign to each of these angles *coinciding with the oriented volume sign* of the tetrahedron to which the angle belongs. Consider a certain edge BC in the triangulation, and let its link contain vertices $A_1, \dots, A_n$, so that the tetrahedra $A_1A_2BC, \dots, A_nA_1BC$ are situated around BC. With our definition for the signs of dihedral angles, one can observe that the algebraic sum of all angles at the edge BC is a multiple of 2π, if these angles are calculated according to the usual formulas of Euclidean geometry, starting from *given coordinates of vertices* $A_1, \dots, A_n$, B and C.

Here is the method we want to use to effectively compute these dihedral angles. Given the coordinates of vertices, we calculate all the *edge lengths* in tetrahedra $A_1A_2BC, \dots, A_nA_1BC$ and the signs of all tetrahedron volumes, and then we calculate dihedral angles from the edge lengths. Suppose now that we have slightly, but otherwise arbitrarily, changed the edge lengths. Each separate tetrahedron $A_1A_2BC, \dots, A_nA_1BC$ remains still a Euclidean tetrahedron, but the algebraic sum of their dihedral angles at the edge BC ceases, generally speaking, to be a multiple of 2π. This means that these tetrahedra can no longer be put in $\mathbb{R}^3$ together. In such situation, we call this algebraic sum, taken with the opposite sign, *deficit angle*, or *discrete curvature* (or also *defect* angle, as in paper [Kor01]) at edge BC:

$$\omega_{BC} = -\sum_{i=1}^{n} \varphi_i \bmod 2\pi, \tag{3}$$

where φ_i are the dihedral angles at BC in the n tetrahedra under consideration. Note that the minus sign in Equation (3) is just due to a convention in "Regge calculus" where such deficit angles often appear.

3.2. **Infinitesimal deformations.** To build our acyclic Complex (5) in Section 4, we need only *infinitesimal* deficit angles arising from infinitesimal deformations of edge lengths in the neighborhood of a "flat" case, where all ω vanish. Nevertheless, it is convenient for us at this moment to think of edge length deformations and corresponding deficit angles as small but finite values.

Let the edge lengths in our tetrahedra $A_1A_2BC, \dots, A_nA_1BC$ be slightly deformed with respect to the flat case $\omega_{BC} = 0$. We can introduce a Euclidean coordinate system in tetrahedron A_1A_2BC. Then, it can be extended to the tetrahedron A_2A_3BC through their common face A_2BC. Continuing in this way, we can go around the edge BC and return in the initial tetrahedron A_1A_2BC, obtaining thus a new coordinate system in it. The transformation from the old system to the new one is given by an element of the group SO(3) which is a rotation around the edge BC through the angle ω_{BC} in a proper direction.

Consider now the vertex B and all the edges that end in it; we call them $BC_1, \dots, BC_m$. Consider a closed path starting inside a tetrahedron having B as one of its vertices, then going through one of its faces (but not touching the *edges*) into a neighboring tetrahedron, which is also supposed to have B as one of its vertices, and so forth until the path returns to its initial point. Dragging a Euclidean coordinate system along such a path, we obtain an element of SO(3), similarly to what we have done in the previous paragraph (note that the vertex B plays the role of the origin of coordinates).

Lemma 2. *If the mentioned closed path can be contracted into a point continuously and in such way that it does not intersect any edge at any moment, then the element of* SO(3) *corresponding to it is the identical transformation.*

Proof. The proof of this statement is completely evident; we have presented it as a separate lemma because it is, nevertheless, quite important in what follows. □

After these generalities, consider what this gives in the situation of infinitesimal curvatures, where all the mentioned elements of SO(3) can be thought of as identical transformation plus elements of the Lie algebra $\mathfrak{so}(3)$. Denote $\vec{e}_{C_iB}$ the unit vector pointing in the direction of an edge C_iB, that is, $\vec{e}_{C_iB} = 1/l_{C_iB} \cdot \overrightarrow{C_iB}$.

Lemma 3. *If infinitesimal deficit angles $d\omega_{C_iB}$ are obtained from infinitesimal deformations of length of edges in a triangulation with respect to the flat case, then*

$$\sum_{i=1}^{m} \vec{e}_{C_iB} \, d\omega_{C_iB} = \vec{0}, \tag{4}$$

where index i numbers all vertices joined with vertex B by edges.

Proof. Lemma 3 is just an infinitesimal version of Lemma 2; Equality (4) is really an equality in algebra $\mathfrak{so}(3)$ which we identify as a vector space with $\mathbb{R}^3$. □

4. The acyclic complex and the invariant

4.1. **Definitions.** Consider the following chain of linear spaces and linear mappings:

$$0 \longrightarrow \begin{pmatrix} dx \\ \text{except} \\ A,B,C,D \end{pmatrix} \xrightarrow{f_2} \begin{pmatrix} dl \\ \text{except the} \\ \text{edges of two} \\ ABCD \end{pmatrix} \xrightarrow{f_3 = f_3^{\mathrm{T}}} \begin{pmatrix} d\omega \\ \text{except the} \\ \text{edges of two} \\ ABCD \end{pmatrix}$$
$$\xrightarrow{f_4 = f_2^{\mathrm{T}}} \begin{pmatrix} \bigoplus \mathfrak{so}(3) \\ \text{except} \\ A,B,C,D \end{pmatrix} \longrightarrow 0. \qquad (5)$$

Here is the detailed description of the vector spaces in the chain Complex (5):

- the first vector space, denoted "(dx except A,B,C,D)", is the vector space of differentials of coordinates of all vertices except A,B,C,D,
- the second vector space, denoted "(dl except the edges of two $ABCD$)", consists of differentials of edge lengths for all edges except those depicted in Figures 1 and 2,
- similarly, the third vector space, denoted "($d\omega$ except the edges of two $ABCD$)", consists of differentials of *deficit angles* corresponding to the same edges,
- the last vector space is a direct sum of copies of the Lie algebra $\mathfrak{so}(3)$ corresponding to the same vertices in the triangulation as in the first space.

Before giving the detailed definitions of mappings f_2, f_3 and f_4, we give some comments.

(i) We use the notations "f_2" and "f_3" to make them consistent with other papers on the subject, such as, e.g. [Mar04, Mar]. So, the reader must not be surprised with not finding any "f_1" in this paper.
(ii) There is a natural basis in each of the vector spaces; it is determined up to an ordering of the vertices in the first and fourth spaces, and up to an ordering of the edges in the second and thirs spaces. Thus, the elements of vector spaces are identified with column vectors, while mappings — with matrices.

(iii) The superscript T means matrix transposing; the equalities over the arrows in Complex (5) will be proved soon after we define the mappings f_2, f_3 and f_4.

For example, the first vector space consists of columns of the kind

$$(dx_{E_1}, dy_{E_1}, dz_{E_1}, \dots, dx_{E_N}, dy_{E_N}, dz_{E_N})^{\mathrm{T}},$$

where $E_1, \dots, E_N$ are all the vertices in the triangulation except A, B, C, D. Note also that the special role of the edges depicted in Figures 1 and 2, announced in the Introduction, is by now reduced to the fact that they simply do not take part in forming the second and third linear spaces in Complex (5). Nevertheless, we will see in Section 5 that they can play a more important role as well.

Here are the definitions of the mappings in the chain Complex (5):

- the definition of the mapping f_2 is obvious: if we slightly change the coordinates of vertices, then the edge lengths will also slightly change according to the formula

$$l_{MN} = \sqrt{(x_N - x_M)^2 + (y_N - y_M)^2 + (z_N - z_M)^2}, \tag{6}$$

  where M and N are two vertices, $x_M, \dots, z_N$ — their coordinates, and l_{MN} — the length of edge MN,
- the mapping f_3 also goes according to Euclidean geometry, although the explicit formulas are more complicated, see [Kor01, KM02] for some of them,
- for the mapping f_4, the element of the Lie algebra corresponding to a given vertex, arising from given curvatures $d\omega$ due to f_4, is by definition given by the left-hand side of formula (4) (where, clearly, the vertex in consideration is substituted in place of "B").

Theorem 4. *Sequence (5) is an algebraic complex, i.e., the composition of two successive maps is zero.*

Proof. First, we prove that $f_3 \circ f_2 = 0$. This is obvious from geometric considerations. Indeed, the edge length changes caused by changes of vertex coordinates give no deficit angles, because the whole picture (vertices and edges) does not go out of a Euclidean space $\mathbb{R}^3$.

Second, the equality $f_4 \circ f_3 = 0$ is simply a reformulation of Lemma 3. □

Complex (5) can be called a *complex of infinitesimal geometric deformations.* Its interesting additional property is a sort of "symmetry" and is presented in the following Theorem.

Theorem 5. *The matrices of mappings in Complex (5) satisfy the following symmetry properties:*

$$f_3 = f_3^{\mathrm{T}}, \qquad f_4 = f_2^{\mathrm{T}}. \tag{7}$$

Proof. The well–known Schläfli differential identity for a Euclidean tetrahedron reads:

$$\sum_{i=1}^{6} l_i \, d\varphi_i = 0$$

for any infinitesimal deformations (l_i are edge lengths in the tetrahedron, and φ_i are dihedral angles at edges). Hence, it follows that

$$\sum_a l_a \, d\omega_a = 0; \tag{8}$$

here and below in this proof a runs over all edges in the triangulation.

Consider the quantity $\Phi = \sum_a l_a \omega_a$ as a function of the lengths l_a, and write the following indentity for it:

$$\frac{\partial^2 \Phi}{\partial l_b \partial l_c} = \frac{\partial^2 \Phi}{\partial l_c \partial l_b}, \tag{9}$$

where b and c are some edges. It is easy to see that Equation (8) together with Equation (9) yield $\partial\omega_b/\partial l_c = \partial\omega_c/\partial l_b$, which is nothing but the first equality in (7).

As for the second equality in Equation (7), it can be proved by a direct writing out of matrix elements, i.e., the relevant partial derivatives. For the mapping f_2, one has to differentiate the Relation (6), and for f_4 — use the left-hand side of Formula (4). □

An acyclic complex possessing the symmetry of the type described in Theorem 5 is sometimes called a *de Rham complex*.

Conjecture. *There are many enough interesting cases where Complex (5) turns out to be acyclic.*

At this time, we cannot make this conjecture more precise. Instead, we will present some relevant examples below in Sections 6–9.

Convention. From now on, we assume that we are working with an *acyclic complex*.

4.2. **Reidemeister torsion and the invariant.** As Complex (5) is supposed to be acyclic, we associate to it its *Reidemeister torsion* given by

$$\tau = \frac{(\text{minor } f_2)^2}{\text{minor } f_3} = \frac{(\det f_2|_{\overline{\mathcal{C}}})^2}{\det f_3|_{\mathcal{C}}}. \tag{10}$$

The letter $\mathcal{C}$ denotes a maximal subset of edges (remember that the edges depicted in Figures 1 and 2 have been already withdrawn) for which the corresponding *diagonal* minor of f_3 does not vanish. We also write this minor in a more precise way as $\det f_3|_{\mathcal{C}}$, where $f_3|_{\mathcal{C}}$ is the submatrix of f_3 whose *rows and columns* correspond to the edges in $\mathcal{C}$. The set $\overline{\mathcal{C}}$ is the complement of $\mathcal{C}$ in the set of all edges except those depicted in Figures 1 and 2, and $f_2|_{\overline{\mathcal{C}}}$ is the submatrix of f_2 whose *rows* correspond to the edges in $\overline{\mathcal{C}}$.

Remark 2. As it is known (see monograph [Tur01]), usually Reidemeister torsion is defined up to a sign, so that special measures must be taken for its "sign-refining". This sign is changed when we change the order of basis vectors in any of the vector spaces. In our case, however, this is not a problem: due to the symmetry proved in Theorem 5, we can choose our torsion in the form (10) where the numerator is a square and the denominator is a *diagonal* minor. Both thus do not depend on the order of basis vectors.

Remark 3. One can notice that our Formulas (10) and (11) are much the same as Formulas (4) and (5) in paper [Kor02], where just three–dimensional manifolds without knots are under consideration.

Theorem 6. *If Complex (5) is acyclic for some pseudotriangulation of the manifold M with a chain of two tetrahedra in it of the kind described in Section 2, then it remains acyclic after any relative Pachner move $2 \leftrightarrow 3$ or $1 \leftrightarrow 4$ (i.e. moves not involving the tetrahedra $ABCD$). Moreover, the following value remains unchanged under such moves:*

$$I = \tau \, \frac{\prod'_{\text{edges}} l^2}{\prod'_{\text{tetrahedra}} (-6V)} \, (6V_{ABCD})^4. \tag{11}$$

Here $\prod'_{\text{edges}} l^2$ means the product of squared lengths for all edges except those depicted in Figures 1 and 2; $\prod'_{\text{tetrahedra}}(-6V)$ means the product of all tetrahedron volumes multiplied by (-6) except two tetrahedra $ABCD$ shown in Figures 1 and 2, and τ is the torsion of Complex (5) given by Formula (10).

Remark 4. Invariant I in Equation (11) can depend a priori on the geometry of the tetrahedron $ABCD$. It turns out that, with the multiplier $(6V_{ABCD})^4$ introduced in Formula (11), the invariant is just a number at least in the examples considered below in Sections 6–9.

Proof of Theorem 6. Suppose we are doing a $2 \to 3$ Pachner move: two adjacent tetrahedra $MNPQ$ and $RMNP$ are replaced by three tetrahedra $MNRQ$, $NPRQ$ and $PMRQ$, see Figure 3. Thus, a new edge QR appears in the triangulation and the following relation holds:

$$\frac{\partial \omega_{QR}}{\partial l_{QR}} = -\frac{l_{QR}^2}{6} \frac{V_{MNPQ} V_{RMNP}}{V_{MNRQ} V_{NPRQ} V_{PMRQ}}. \tag{12}$$

Observe that this is the most important formula allowing us to construct manifold invariants based on three–dimensional Euclidean geometry. In our previous papers, we must admit that we not always cared for the *signs* in such formulas; Formula (12), with the right sign, can be found as Formula (2.6) in [KM02].

Note that Equation (12) gives the ratio between the "new" minor f_3 and the "old" minor f_3 in Formula (10); this statement coincides with Equation (18) in [Kor01], where the reader can find a proof of this statement.

As a consequence, Equation (12) is also the inverse ratio between the "new" and "old" torsions given by Formula (10). The comparison of that with Formula (11) proves that I does not change under a $2 \to 3$ Pachner move. Of course, this also proves the invariance of I under the inverse move $3 \to 2$.

Now we consider a $1 \to 4$ Pachner move. It means that a tetrahedron $MNPQ$ is divided into four tetrahedra by adding a new vertex R inside it, as in Figure 4. Hence, three new components to the vectors in the space (dx) — the first vector space in Sequence (5) — are added, namely the differentials dx_R, dy_R and dz_R of coordinates of vertex R. In the same way, four new components to the vectors in the space (dl) — the second vector space in Sequence (5) — are added, namely $dl_{MR}, dl_{NR}, dl_{PR}$ and dl_{QR}. We add the edge QR to the set $\mathcal{C}$, then MR, NR and PR are added to $\overline{\mathcal{C}}$. The minor of f_2 thus gets multiplied by

$$\frac{dl_{MR} \wedge dl_{NR} \wedge dl_{PR}}{dx_R \wedge dy_R \wedge dz_R} = \frac{6V_{MNPR}}{l_{MR} l_{NR} l_{PR}} \tag{13}$$

(compare Formulas (31) and (32) in [Kor01]). Due to the same considerations, the minor of f_3 gets multiplied by the very same factor as in Equation (12) as we have already written out in the case of a $2 \to 3$ Pachner move. Comparing Equations (10), (11), (12) and (13), we see that our value I does not change under a $1 \to 4$ Pachner move, as well as, under the inverse move. □

Remark 5. Observe that the acyclicity of our Complex (5) is preserved under the Pachner moves: this follows from the fact that no minors considered in our proof vanished (or turned into infinity).

5. How to change the framing

Just as Pachner moves are elementary rebuildings of a triangulation of a *closed* manifold, *shellings* and inverse shellings are elementary rebuildings of a triangulation of a manifold *with boundary*, see [Lic99, section 5]. A topological field theory dealing with triangulated manifolds must answer the question what happens with an invariant like our invariant I under shellings.

While we leave a general answer to this questions to further papers, we will explain in this section how some shellings on the toric boundary of our manifold "M minus two tetrahedra" correspond to changing the framing of the knot determined by these two tetrahedra. We also show what happens with matrices f_3 and f_2 from Complex (5) under these shellings. A result which will be used in Section 9.

It is enough to learn how to change the knot framing by one–half of a revolution. We can achieve this if we manage to "turn inside out" one of the tetrahedra $ABCD$ in Figures 1 and 2, e.g., in the way shown in Figure 6.

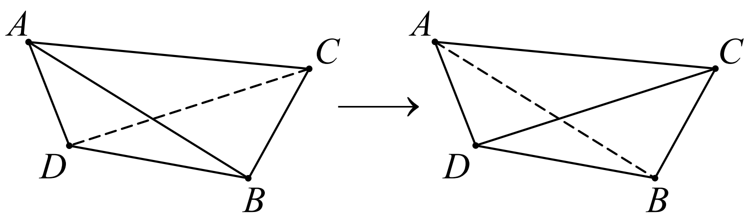

FIGURE 6. Turning a tetrahedron inside out, thus changing the framing by $1/2$

Remark 6. Of course, the framing can be changed in other direction similarly. In this case, we should first draw the left–hand–side tetrahedron in Figure 6 as viewed from another direction, so as the diagonals of its projection are AC and BD, instead of AB and CD in Figure 6. Then we replace the dashed "diagonal" with the solid one and vice versa.

Return to Figure 6. In order to be able to glue the "turned inside out" tetrahedron back into the triangulation, we can glue to it two more tetrahedra $ABCD$: one to the front and one to the back. Before explaining this in more detail, remember that we are considering a *pseudo*triangulation, which roughly speaking means that we can think of our tetrahedra as flexible and elastic. So, we glue the same tetrahedron as drawn in the left-hand side of Figure 6, to the *two* "front" faces, ADC and DBC, of the "turned inside out" tetrahedron in the right–hand side of Figure 6, and again the same tetrahedron as in the left–hand side of Figure 6 to the two "back" faces, ABC and ADB (always glueing a vertex to the vertex of the same name). After this, the obtained "sandwich" of three tetrahedra can obviously be glued into the same place which was occupied by the single tetrahedron in the left-hand side of Figure 6.

In such way, it is clear that the manifold "M minus a chain of one of the types in Figures 1 and 2" changes its boundary: two tetrahedra between which we put the right–hand–side tetrahedron in Figure 6 are glued to it in the very same way as is used for shellings (however, note that in contrast with paper [Lic99], we are considering a *pseudo*triangulation). How will the invariant I change? Of course, the product of tetrahedron volumes in (11) will be multiplied by the squared volume of $ABCD$, and the product of edge lengths will be multiplied by l_{AB} and l_{CD}, because we have added two tetrahedra $ABCD$ to our pseudotriangulation, and the edges AB and CD of the "initial" tetrahedron in the left-hand side of Figure 6 changed their status from being inner to lying on the boundary.

To describe the change of matrix f_3 under the change of framing by $1/2$ going this way, it is first convenient to introduce matrix F_3, which consists by definition of *all* partial derivatives $(\partial\omega_a/\partial l_b)$, *including the edges that belong to the distinguished tetrahedra* in Figures 1 and 2. Thus, f_3 is a submatrix of F_3. Then we introduce a "normalized" version of matrix F_3, denoted G_3, as follows:

$$G_3 = 6V_{ABCD}\,\mathrm{diag}(l_1^{-1},\ldots,l_{N_1}^{-1})F_3\,\mathrm{diag}(l_1^{-1},\ldots,l_{N_1}^{-1}). \tag{14}$$

Here N_1 is the total number of edges in the triangulation of the manifold M (and it changes when we add new edges). Just as f_3, matrices F_3 and G_3 are symmetric.

Now we describe what happens with G_3 when we change the framing. We represent the "initial" G_3 in a block form where the last row and the last column correspond to the edge CD, and the next to last row and column to the edge AB:

$$G_3 = \begin{pmatrix} K & L^{\mathrm{T}} \\ L & \begin{matrix} \alpha & \beta \\ \beta & \gamma \end{matrix} \end{pmatrix}. \tag{15}$$

Here α, β and γ are real numbers, K is an $(N_1-2)\times(N_1-2)$ block, and L is a $2\times(N_1-2)$ block.

Recall that we have chosen a consistent orientation for all tetrahedra in the triangulation, which means, for every tetrahedron, an ordering of its vertices up to even permutations. The "initial" tetrahedron in the left–hand side of Figure 6 thus can have either orientation $ABCD$ or $BACD$. When we replace this tetrahedron by a "sandwich" as described above, the innermost tetrahedron in the "sandwich" acquires the opposite orientation compared to the initial tetrahedron.

Theorem 7. *After the change of framing which adds two new edges AB and CD to the pseudotriangulation in the way described above, matrix G_3*

is changed to a new matrix denoted $(G_3)_{\text{new}}$ *and which admits the following form:*

$$(G_3)_{\text{new}} = \begin{pmatrix} K & L^{\mathrm{T}} & & \mathbf{0} & \\ L & \begin{matrix} \alpha & \beta-\epsilon \\ \beta-\epsilon & \gamma \end{matrix} & & \begin{matrix} 0 & \epsilon \\ \epsilon & 0 \end{matrix} & \\ \mathbf{0} & \begin{matrix} 0 & \epsilon \\ \epsilon & 0 \end{matrix} & & \begin{matrix} 0 & -\epsilon \\ -\epsilon & 0 \end{matrix} & \end{pmatrix}. \tag{16}$$

Here $\epsilon = -1$ *if the initial tetrahedron has the orientation* $BACD$ *and* $\epsilon = 1$ *if it has the orientation* $ABCD$*. Moreover, the two new rows and the two new columns corresponding to the new edges* AB *and* CD *(belonging to the right–hand side tetrahedron in Figure 6) are added as two last rows and columns of the matrix in Formula (16).*

Proof. The normalization (16) of matrix G_3 has been chosen keeping in mind the formula for the partial derivative of a dihedral angle in a tetrahedron w.r.t. the length of the *opposite* edge, with other edge lengths fixed:

$$\frac{\partial \varphi_a}{\partial l_b} = \frac{l_a l_b}{6V} \tag{17}$$

(compare with [Kor01, Formula (3)]), where φ_a is the dihedral angle at edge a, and V is tetrahedron volume. As the angles φ enter in a deficit angle with a minus sign (Formula (3)), the derivatives like (17) contribute to the elements of matrix G_3 as -1 if the orientation of the corresponding tetrahedron is $ABCD$, and as $+1$ if it is $BACD$. This, first, explains why ϵ is subtracted from the matrix element β when the "initial" edges AB and CD cease to belong to the same tetrahedron. Second, it explains the appearance of $\pm\epsilon$ in the last two rows and columns of $(G_3)_{\text{new}}$. It remains to explain why the other new matrix elements vanish, e.g., why

$$\frac{\partial \omega_{AC}}{\partial l_{AB_{\text{new}}}} = 0, \qquad \frac{\partial \omega_{AB_{\text{new}}}}{\partial l_{AB_{\text{new}}}} = 0 \quad \text{and so on,} \tag{18}$$

and why some other elements in G_3 do not change while, seemingly, the triangulation change has touched them.

The first equality in Equation (18) is due to the fact that $l_{AB_{\text{new}}}$ influences two dihedral angles which enter in ω_{AC}; these angles belong to two tetrahedra $ABCD$ which differ only in their orientations and thus sum up to an identical zero. A similar explanation works for the second equality in (18) as well. Moreover, a similar reasoning shows that, although new summands are added to some elements in G_3 like $\partial\omega_{AC}/\partial l_{AD}$, these summands cancel each other because they belong to tetrahedra with opposite orientations. □

From matrix $(G_3)_{\mathrm{new}}$, we can obtain the new matrix F_3, and then take its relevant submatrix as new f_3. As for the matrix f_2 in Complex (5), it will just acquire two new rows whose elements, like all elements in f_2, are obtained by differentiating relations of type (6).

6. Lens spaces: their triangulations and framed unknots in them

We now turn to applications of our ideas to concrete some manifolds with a framed knot in it. A rich and historical set of examples is supplied by *lens spaces* with their pseudotriangulation that arises naturally from the representation of a lens space as a bipyramid with its faces identified according to some rule described below.

6.1. **Generalities on lens spaces and their triangulations.**

Let p, q be two coprime integers such that $0 < q < p$. We identify S^3 with the subset $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ of $\mathbb{C}^2$. The lens space $L(p, q)$ is defined as the quotient manifold $S^3/\sim$, where $\sim$ denotes the action of the cyclic group $\mathbb{Z}_p$ on S^3 given by:

$$\zeta \cdot (z_1, z_2) = (\zeta z_1, \zeta^q z_2), \quad \zeta = e^{2\pi i/p}. \tag{19}$$

As a consequence the universal cover of lens spaces is the three–dimensional sphere S^3 and

$$\pi_1\big(L(p, q)\big) = H_1\big(L(p, q)\big) = \mathbb{Z}_p. \tag{20}$$

Remark 7. In Formula (19), the group $\mathbb{Z}_p$ is understood as a multiplicative group of roots of unity of degree p. Below, it will be more convenient for us to consider it as an additive group consisting of integers between 0 and $p - 1$ whose addition is understood modulo p.

Now we describe triangulations of $L(p, q)$ which will be used in our computations. Consider the bipyramid of Figure 7, which contains p vertices B and p vertices C. The lens space $L(p, q)$ is obtained by gluing the upper half of its surface to the lower half, the latter having been rotated around the vertical axis through the angle of measure $2\pi q/p$ in such way that every "upper" triangle BCD is glued to some "lower" triangle BCD (the vertices of the same names are identified).

6.2. **Knots in lens spaces.**

A generator of the fundamental group can be represented, e.g., by some broken line BCB (the two end points B are different) lying in the equator of the bipyramid. We assume that a generator chosen in such way corresponds to the element $1 \in \mathbb{Z}_p$ under the identification of Equation (20).

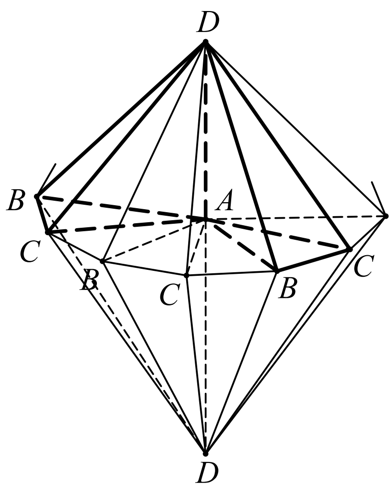

FIGURE 7. A chain of two tetrahedra in a lens space

The boldface lines (solid and dashed) in Figure 7 single out two *identically oriented* tetrahedra $ABCD$ which form a chain exactly like the one in Figure 1. Indeed, one can see in Figure 7 that they have a common edge AD, and their edges BC are identified as well according to the construction of the lens space. Going along the chain of tetrahedra in Figure 7 (e.g., along the way BAB) corresponds, under the agreement of the previous paragraph, to the element $2 \in \mathbb{Z}_p$ (or to $-2 \in \mathbb{Z}_p$, if we go in the opposite direction). It is clear that one can also choose a pair of tetrahedra corresponding to *any nonzero* element from $H_1\big(L(p,q)\big) = \mathbb{Z}_p$.

A knot in $L(p,q)$ determined by a tetrahedron chain of the kind of Figure 7 (and corresponding to any nonzero element of the first homology group), i.e., going along a line like BAB, can be called, somewhat loosely, an "unknot" in $L(p,q)$. It differs from any other conceivable knot, going along which gives the same element of $H_1\big(L(p,q)\big)$, in its "minimal knottedness" in the following sense: the full preimage of this knot in the universal cover of space $L(p,q)$, i.e., sphere S^3, being decomposed in a connected sum of simple knots, contains the *smallest number of summands.* Indeed, the line BAB is equivalent, as a knot, to the segment of the straight line joining the two points B; if, on the other hand, we tie a nontrivial knot on this segment, there will appear p new summands in the full preimage (in the sense of connected summation) equivalent to this nontrivial knot.

7. Lens spaces: the structure of matrix $(\partial\omega_a/\partial l_b)$

The pseudotriangulation of a lens space, described in the previous section, does not contain any vertices besides A, B, C and D. It follows then that the algebraic complex (5), corresponding to such pseudotriangulation, is reduced to a single mapping f_3, that is, it takes form

$$0 \longrightarrow \begin{pmatrix} dl \\ \text{except the} \\ \text{edges of two} \\ ABCD \end{pmatrix} \xrightarrow{f_3} \begin{pmatrix} d\omega \\ \text{except the} \\ \text{edges of two} \\ ABCD \end{pmatrix} \longrightarrow 0\,. \tag{21}$$

This complex is acyclic provided $\det f_3 \neq 0$.

As we have explained in Section 5, it makes sense to consider the matrix F_3 which consists, by definition, of the partial derivatives of *all* deficit angles with respect to *all* edge lengths in the pseudotriangulation of the lens space and of which f_3 is a submatrix. Moreover, it makes sense to consider the "normalized" version of F_3, i.e., matrix G_3 defined by the Equality (14).

Matrix G_3 has many zero entries. This can be explained in one of two ways: either the corresponding derivative $\partial\omega_a/\partial l_b$ vanishes because the edges a and b do not belong to the same tetrahedron, or the cause is like that explained in the proof of Theorem 7, compare Formula (18). Namely, if two edges a and b belong to the same two–dimensional face (this includes, in particular, the case $a = b$), then the summands in the derivative

$$\frac{\partial\omega_a}{\partial l_b} = -\sum_i \frac{\partial(\varphi_a)_i}{\partial l_b},$$

where i numbers the tetrahedra around edge a, can be grouped in pairs for which the two derivatives $\partial(\varphi_a)_i/\partial l_b$ are equal in absolute value but differ in signs, because the two corresponding tetrahedra have opposite orientations.

It will be convenient for us to denote the pseudotriangulation edges by indication of origin and end vertices of a given edge. However, in order to stress that some different edges may have the same origin and end vertices, we will use indices $1, 2, \ldots, p$, as indicated in Figure 8. For example, as one can see from this figure, there exist p different edges AB, and each of them is equipped with an index from 1 to p. So, we denote by $(AB)_n$ the edge AB equipped with index n.

To describe the structure of matrix G_3, we introduce the following ordering on the set of all edges in pseudotriangulation:

$$(AB)_1, \ldots, (AB)_p,\ (CD)_1, \ldots, (CD)_p,\ (AC)_1, \ldots, (AC)_p,\ (BD)_1, \ldots, (BD)_p,$$
$$(AD)_1, (AD)_2,\ (BC)_1, (BC)_2.$$

In this way, we put in order the basis vectors in spaces (dl) and $(d\omega)$. The order of matrix G_3 is $(4p + 4) \times (4p + 4)$, and with respect to he preceding

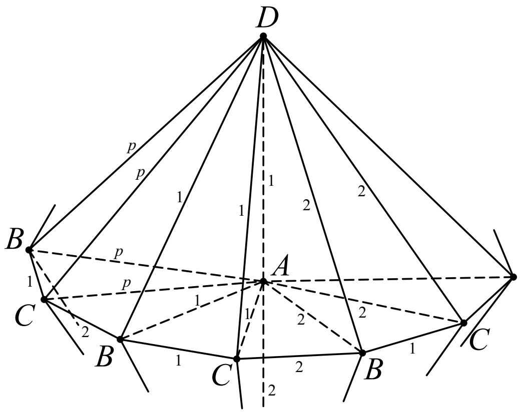

FIGURE 8. To the explanation of the structure of matrix G_3

ordered basis, G_3 has the following block structure:

$$G_3 = \left(\begin{array}{cc|cc|cc} \mathbf{0}_p & S_1 & & & & \\ S_1^T & \mathbf{0}_p & & & & \\ \hline & & \mathbf{0}_p & S_2 & & \\ & & S_2^T & \mathbf{0}_p & & \\ \hline & & & & \mathbf{0}_2 & S_3 \\ & & & & S_3^T & \mathbf{0}_2 \end{array}\right), \tag{22}$$

where S_1, S_2 are $p \times p$ submatrices and S_3 is a 2×2 submatrix. Here and below the empty spaces in matrices are of course occupied by zeroes.

We now describe the structure of the three blocks S_1, S_2 and S_3 respectively.

(i) The i–th row of S_1 consists of the partial derivatives $\partial\omega_{(AB)_i}/\partial l_{(CD)_j}$, and with the help of Figure 8, we may conclude that there exist exactly four nonzero entries in each row, namely:

$$c \cdot \frac{\partial\omega_{(AB)_i}}{\partial l_{(CD)_i}}, \quad c \cdot \frac{\partial\omega_{(AB)_i}}{\partial l_{(CD)_{i-1}}}, \quad c \cdot \frac{\partial\omega_{(AB)_i}}{\partial l_{(CD)_{i-q}}}, \quad c \cdot \frac{\partial\omega_{(AB)_i}}{\partial l_{(CD)_{i-q-1}}}, \tag{23}$$

where

$$c = \frac{6V_{ABCD}}{l_{AB} l_{CD}}.$$

Here, all indices change cyclicly from 1 to p, i.e., for instance, $0 \equiv p$, $-1 \equiv p-1$, and so forth. It is convenient to choose the orientation of the four tetrahedra in Figures 1 and 2 as $BACD$, then the expressions in Equation (23) turn respectively into

$$1, \quad -1, \quad -1, \quad 1,$$

according to Equations (17) and (3). Moving further along these lines, we obtain the following formula for S_1:

$$S_1 = \mathbf{1}_p - E - E^q + E^{q+1} = (\mathbf{1}_p - E^q)(\mathbf{1}_p - E), \tag{24}$$

where $\mathbf{1}_p$ is the identity matrix of size $p \times p$, and

$$E = \begin{pmatrix} 0 & & \dots & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}. \tag{25}$$

(ii) Similarly,

$$S_2 = (\mathbf{1}_p - E^q)(\mathbf{1}_p - E^{-1}). \tag{26}$$

(iii) Finally, one can verify that

$$S_3 = p \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{27}$$

8. Lens spaces: invariant for the "simplest" framing

8.1. **Formulation of the result.** Consider the simplest case when a framed knot is determined directly by a tetrahedron chain of the type depicted in Figure 7 (with the understanding that the angular distance between the two tetrahedra can be different). In this situation and according to Formulas (10) and (11) and the form of the acyclic Complex (21), the invariant comes out to be as:

$$I = \frac{(6V_{ABCD})^4 \prod'_{\text{edges}} l^2}{\det f_3 \prod'_{\text{tetrahedra}} (-6V)} = \frac{1}{\det g_3}, \tag{28}$$

where g_3 is the submatrix of G_3 consisting of the same rows and columns of which consists f_3 as a submatrix of F_3.

Recall that Figure 7 shows an "unknot going along the element $2 \in \mathbb{Z}_p = H_1\big(L(p,q)\big)$" in the sense that the two distinguished tetrahedra are turned into each other under a rotation through angle $2 \cdot \frac{2\pi}{p}$; similarly, an "unknot going along the element $n \in \mathbb{Z}_p = H_1\big(L(p,q)\big)$" is determined by two tetrahedra which differ in a rotation through angle $n \cdot \frac{2\pi}{p}$. For a different basis element in H_1, this number n would change, but we are considering the lens space $L(p,q)$ as constructed in a fixed way from the given bipyramid in Figure 7.

We also identify $n \in \mathbb{Z}_p$ with one of positive integers $1, \ldots, p-1$ (of course, $n \neq 0$). One can see that matrix g_3, for a given n, can be obtained by taking away from G_3 the rows and columns number n, p, $p+n$, $2p$, $2p+n$, $3p$, $3p+n$, $4p$, $4p+1$, and $4p+3$. Let $\tilde{S}_1$ (resp. $\tilde{S}_2$) denote the $(p-2) \times (p-2)$ matrix obtained by removing the n–th and p–th rows and columns from the matrices S_1 (resp. S_2). We set:

$$s_n = \det \tilde{S}_1, \qquad t_n = \det \tilde{S}_2.$$

Also, let $\tilde{S}_3$ denote the matrix obtained by removing the first row and column from the matrix S_3, that is, $\tilde{S}_3 = (p)$.

Let $I_n\big(L(p,q)\big)$ denote the invariant of the framed knot in $L(p,q)$ defined by a tetrahedron chain like in Figure 7, but with the angular distance $n \cdot \frac{2\pi}{p}$ between the tetrahedra. With this notation we have the following explicit result for lens spaces.

Theorem 8. *The invariant $I_n\big(L(p,q)\big)$ is explicitly given by the following equation:*

$$I_n\big(L(p,q)\big) = -\frac{1}{s_n^2\, t_n^2\, p^2}, \tag{29}$$

in which the values s_n and t_n are given by:

$$s_n = nq_n^* - p\,\nu_n, \tag{30}$$

$$t_n = p - s_n = p\,(\nu_n + 1) - nq_n^*, \tag{31}$$

where

$$\nu_n = \frac{1}{p} \sum_{k=0}^{p-1} \frac{1-\zeta^{k(1-n)}}{1-\zeta^k}\, \frac{1-\zeta^{kq(q_n^*-1)}}{1-\zeta^{-kq}}, \quad \zeta = e^{2\pi i/p} \tag{32}$$

and $q_n^* \in \{1, \ldots, p-1\}$ is such that $qq_n^* \equiv n (\mathrm{mod} p)$.

Remark 8. As we will prove, the values ν_n, defined in Equation (32), are integers too and they belong to $\{0, \ldots, n-1\}$. So, we have the congruences $s_n \equiv nq_n^* (\mathrm{mod} p)$ and $t_n \equiv -nq_n^* (\mathrm{mod} p)$.

Remark 9. The indeterminacies $\frac{0}{0}$ arising in Equation (32) when $k=0$ are expanded as the limits of the respective expressions taken when $k \to 0$.

Proof of Theorem 8. Equation (29) is directly deduced from the block structure (22) of matrix G_3 and Equation (28). So, it remains only to find the values s_n and t_n.

(1) We first prove the Formula (30). We use the factorization of matrix S_1 given by Equation (24) in order to simplify the matrix $\tilde{S}_1$ with the

help of certain sequence of elementary transformations preserving the determinant.

Recall that matrix $\tilde{S}_1$ is obtained from matrix

$$S_1 = (\mathbf{1}_p - E^q)(\mathbf{1}_p - E) \tag{33}$$

by taking away n–th and p–th columns and rows. This means that $\tilde{S}_1$ can be obtained also as a product like (33), but with the corresponding rows withdrawn from matrix $(\mathbf{1}_p - E^q)$, and corresponding columns withdrawn from matrix $(\mathbf{1}_p - E)$. Note that below, when we are speaking of row/column numbers in matrix $\tilde{S}_1$, we mean the numbers that these rows/columns had in S_1, *before* we have removed anything from it.

So, here are our elementary transformations. In matrix $(\mathbf{1}_p - E^q)$, for each integer k from 1 to $q_n^* - 1$, we add the (kq)–th row to the $(kq + q)$–th row (numbers modulo p). In matrix $(\mathbf{1}_p - E)$, we first add the $(p-1)$–th column to the $(p-2)$–th one, then we add the $(p-2)$–th column to the $(p-3)$–th one and so forth omitting the pair of column numbers $n-1$ and $n+1$. Then, the resulting matrix has a determinant is equal to $s_n = \det \tilde{S}_1$, and admits the following structure:

$$\begin{pmatrix} \mathbf{1}_{p-2} & \mathbf{c}_n & \mathbf{c}_p \end{pmatrix} \begin{pmatrix} \mathbf{1}_{p-2} \\ \mathbf{r}_n \\ \mathbf{r}_p \end{pmatrix} = \mathbf{1}_{p-2} + \mathbf{c}_n \otimes \mathbf{r}_n + \mathbf{c}_p \otimes \mathbf{r}_p, . \tag{34}$$

Here we have also moved the n–th row in matrix $(\mathbf{1}_p - E^q)$ to the $(p-1)$–th position and the n–th column in matrix $(\mathbf{1}_p - E)$ to the $(p-1)$–th position. The components of column $\mathbf{c}_p$ and row $\mathbf{r}_n$ look like

$$(\mathbf{c}_p)_i = \begin{cases} 1, & i = kq (\operatorname{mod} p), \quad k = 1, \dots, q_n^* - 1, \\ 0, & \qquad\qquad\qquad\quad \text{otherwise}, \end{cases}$$

$$(\mathbf{r}_n)_i = \begin{cases} 1, & i = 1, \dots, n-1, \\ 0, & i = n, \dots, p-2. \end{cases}$$

Moreover, for all i we have

$$(\mathbf{c}_n)_i = 1 - (\mathbf{c}_p)_i, \quad (\mathbf{r}_p)_i = 1 - (\mathbf{r}_n)_i. \tag{35}$$

The matrix $\mathbf{c}_n \otimes \mathbf{r}_n + \mathbf{c}_p \otimes \mathbf{r}_p$ has rank 2, so its eigenvalues are

$$0, \dots, 0, \lambda_1, \lambda_2,$$

where λ_1, λ_2 are the eigenvalues of the following matrix of size 2×2:

$$\begin{pmatrix} \mathbf{r}_n \mathbf{c}_n & \mathbf{r}_n \mathbf{c}_p \\ \mathbf{r}_p \mathbf{c}_n & \mathbf{r}_p \mathbf{c}_p \end{pmatrix}.$$

Therefore, from Equation (34), we can deduce that the determinant of $\tilde{S}_1$ is equal to

$$\det \tilde{S}_1 = s_n = \begin{vmatrix} 1 + \mathbf{r}_n \mathbf{c}_n & \mathbf{r}_n \mathbf{c}_p \\ \mathbf{r}_p \mathbf{c}_n & 1 + \mathbf{r}_p \mathbf{c}_p \end{vmatrix}. \tag{36}$$

Further, using Formula (35) and elementary transformations, we simplify this determinant to

$$s_n = \begin{vmatrix} n & \mathbf{r}_n \mathbf{c}_p \\ p & q_n^* \bmod p \end{vmatrix} = n q_n^* - p\, \mathbf{r}_n \mathbf{c}_p,$$

where the inner product $\mathbf{r}_n \mathbf{c}_p$ is an integer between 0 and $n-1$.

Finally, using the discrete Fourier transform, one can prove that

$$\mathbf{r}_n \mathbf{c}_p = \nu_n = \frac{1}{p} \sum_{k=0}^{p-1} \frac{1 - \zeta^{k(1-n)}}{1 - \zeta^k} \frac{1 - \zeta^{kq(q_n^*-1)}}{1 - \zeta^{-kq}},$$

where $\zeta = e^{2\pi i/p}$.

(2) Quite similarly, we obtain Formula (31).

$\square$

9. Lens spaces: invariant for all framings

9.1. **Formulation of the result.** In the present section, we perform the computation of our invariant for all framings of all unknots in $L(p, q)$.

According to Section 5 we should investigate the change of matrix G_3 under the change of the framing. We assume that we do the first half–revolution exactly as described in Section 5, and the second half-revolution goes in a similar way but with the pair of edges AB, CD replaced by the pair AC, BD, the third half-revolution involves again the pair AB, CD and so on.

Thus, we have to study how the submatrices S_1 and S_2 of G_3 change, because they correspond (according to Formula (22)) to the pairs AB, CD and AC, BD respectively. We think of these matrices as made of the following blocks:

$$S_i = \begin{pmatrix} K_i & M_i \\ L_i & \beta_i \end{pmatrix}, \quad i = 1, 2, \tag{37}$$

where K_i is a $(p-1) \times (p-1)$ matrix, L_i and M_i are row and column of size $p-1$ respectively, and β_i is a real number.

Keep the notation used in Section 5 and in particular the one of Theorem 7. The changes made in matrices S_1 and S_2 follow from Formula (16). When we do the first half–revolution we have $\epsilon = 1$ according to our agreement that the orientation of the tetrahedra in Figures 1 and 2 is $BACD$.

When we do the second half–revolution we have $\epsilon = -1$, because the orientation of the "initial" (or better to say, the innermost in the "sandwich", see Section 5) tetrahedron has changed. Then ϵ takes again the value -1, and so on.

Suppose we have done this way h *half*-revolutions, $h \in \mathbb{N}$. We let $S_1^{(h)}$ (resp $S_2^{(h)}$) denote the matrices obtained from S_1 (resp S_2) according to Formula (16). We get:

$$S_1^{(2m-1)} = S_1^{(2m)} = \begin{pmatrix} K_1 & M_1 & & & & \\ L_1 & \beta_1 - 1 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & -2 & 1 \\ & & & & 1 & -1 \end{pmatrix}, \tag{38}$$

where the total number of (-2)'s is $m-1$, and

$$S_2^{(2m)} = S_2^{(2m+1)} = \begin{pmatrix} K_2 & M_2 & & & & \\ L_2 & \beta_2 + 1 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix}, \tag{39}$$

where the total number of 2's is $m-1$. By definition, $S_1^{(-1)} = S_1^{(0)} = S_1$ and $S_2^{(0)} = S_2$.

In conformity with the notation used in Section 8, we let $\tilde{S}_1^{(2m)}$ denote the matrix obtained by taking away the n–th and the last columns and rows from matrix $S_1^{(2m)}$. Set $s_n^{(2m)} = \det \tilde{S}_1^{(2m)}$. Quite similarly we define $s_n^{(2m+1)} = \det \tilde{S}_1^{(2m+1)}$ and $t_n^{(2m)} = \det \tilde{S}_2^{(2m)}$. The following result gives the value of our invariant for all framings of all unknots in $L(p,q)$.

Theorem 9. *The invariant $I_n^{(r)}\big(L(p,q)\big)$ is given by the following formula:*

$$I_n^{(r)}\big(L(p,q)\big) = \begin{cases} -\dfrac{1}{\big(s_n^{(2m)}\big)^2 \big(t_n^{(2m)}\big)^2 p^2} & \text{if} \quad r = 2m, \\[2ex] -\dfrac{1}{\big(s_n^{(2m+1)}\big)^2 \big(t_n^{(2m)}\big)^2 p^2} & \text{if} \quad r = 2m+1, \end{cases} \tag{40}$$

in which

$$s_n^{(2m)} = (-1)^m (s_n - mp), \tag{41}$$

$$s_n^{(2m+1)} = (-1)^{m+1}(s_n - mp - p), \tag{42}$$

and

$$t_n^{(2m)} = t_n + mp = p - s_n + mp. \tag{43}$$

Proof. From Equation (28), we have two formulas for the invariant:

$$I_n^{(2m)}\big(L(p,q)\big) = -\frac{1}{\big(s_n^{(2m)}\big)^2\big(t_n^{(2m)}\big)^2 p^2} \tag{44}$$

and

$$I_n^{(2m+1)}\big(L(p,q)\big) = -\frac{1}{\big(s_n^{(2m+1)}\big)^2\big(t_n^{(2m)}\big)^2 p^2}. \tag{45}$$

So, what remains is to specify the values of $s_n^{(2m)}$, $t_n^{(2m)}$ and $s_n^{(2m+1)}$.

First of all, we need a lemma concerning matrices S_1 given by Equation (24) and S_2 given by Equation (26). Note that they are degenerate, so they do not have inverse matrices. Instead, we can consider their *adjoint* matrices, whose rank is necessarily not bigger than 1.

Lemma 10. *The adjoint matrix to both matrices S_1 and S_2 has all its elements equal to p.*

Proof of Lemma 10. It is quite easy to see that the adjoint matrix to matrix $\mathbf{1}_p - E^r$, where E is given by Formula (25) and p and r are relatively prime, is a matrix whose all elements are unities. When we take a product like in Equations (24) or (26), the corresponding adjoint matrices are also multiplied (this can be seen at once if we think of matrices S_1 and S_2 and their factors in (24) and (26) as limits of some nondegenerate matrices, keeping in mind that the adjoint to a nondegenerate matrix A is $\det A \cdot A^{-1}$). The product of two $(p \times p)$–matrices whose all elements are 1 is a matrix whose all elements are p. □

Return to the proof of Theorem 9. First, we prove Formula (41). By definition, $s_n^{(0)} = s_n$. Let us find the value

$$s_n^{(2)} = \det \tilde{S}_1^{(2)} = \begin{vmatrix} \tilde{K}_1 & \tilde{M}_1 \\ \tilde{L}_1 & \beta_1 - 1 \end{vmatrix}, \tag{46}$$

where $\tilde{K}_1$ means the matrix K_1 without its n–th row and n–th column; $\tilde{L}_1$ and $\tilde{M}_1$ mean L_1 and M_1 without their n–th entries.

It follows from Lemma 10 that

$$\begin{vmatrix} \tilde{K}_1 & \tilde{M}_1 \\ \tilde{L}_1 & \beta_1 \end{vmatrix} = p. \tag{47}$$

Indeed, the matrix

$$\begin{pmatrix} \tilde{K}_1 & \tilde{M}_1 \\ \tilde{L}_1 & \beta_1 \end{pmatrix}$$

is just the matrix S_1 (compare with Formula (37)) without its n–th row and n–th column, so the determinant (47) is the corresponding element of the matrix adjoint to S_1. Comparing Equations (46) and (47), we get

$$s_n^{(2)} = p - s_n.$$

Let $m > 1$. Using the row (or column) expansion of the determinant of matrix $\tilde{S}_1^{(2m)}$, one can see that a number sequence $s_n^{(2m)}$ is defined by the recurrent condition

$$s_n^{(2m)} = -2s_n^{(2m-2)} - s_n^{(2m-4)}. \tag{48}$$

Besides, as we have just shown, there are the following initial conditions $s_n^{(0)} = s_n$, $s_n^{(2)} = p - s_n$. Hence, using induction on m, we get Formula (41).

As for the Formulas (42) and (43), they are proved in much the same manner. □

Remark 10. One can see that the same formulas hold also for *all* integers m, including the case where $2m < 0$ or $2m + 1 < 0$.

Remark 11. If the value of our invariant (40) turns into infinity, then this means that the Complex (21) is *not acyclic*. However, one can check that, e.g., for $p = 7$ this never happens.

Remark 12. For instance, if we consider the lens space $L(7, 1)$, the invariant given by Equation (40) is enough to distinguish *all* unknots with *all* framings from each other. One can see this from a direct calculation.

10. Discussion of results

Here are some remarks about the results of this paper and possible further directions of research.

(1) Given the results of Theorems 8 and 9, we observe that our invariant, although being just one number, is nontrivial in the case of lens spaces with "unknots" in them. For example, as stated in Remark 12, it is powerful enough to distinguish between all "unknots" with all framings in $L(7, 1)$. So, it is of course interesting to apply it to other situations, and one natural example is nontrivial knots in S^3.

(2) Comparing our geometric torsion and the "usual" Reidemeister torsion, we can now see some difference between them. Our invariant

works more like a quantum invariant in the sense that it does not require any nontrivial representation of the fundamental group (be it the fundamental group of the manifold or of the knot complement). Perhaps, a non-commutative or quantum version of our invariant can be developed with time.

(3) One more direction of research is suggested by the presence of a framed knot in our constructions. As we know, this is usually used for obtaining new closed manifolds by means of a surgery. For example, lens spaces are obtained by surgery on the unknot in S^3 with a certain framing, which is exactly the slope of the surgery. So, the idea is, in a general formulation, to explore more in-depth the behavior of our invariant under surgeries. This can require more research on what happens with our invariants under shellings of a manifold boundary.

(4) As we have mentioned in the Introduction, in the case of a nontrivial representation of a manifold fundamental group or knot group, our invariants appear to be related to the Reidemeister torsion, including even the non-abelian case, see [Mar04] and compare with [Dub05]. As is known [Dub05], the Reidemeister torsion is related to the *volume form* on the character variety of the fundamental group. Note also that, for example [BHKK01], the structure of the character variety of the fundamental group of the manifold obtained by a surgery on a knot is well–known: it can be deduced using the character variety of the knot complement and the slope of the surgery. So, we can try to search for possible links between that and the set of real numbers we obtain using our invariant, guided by a general idea that, in a sense, the representations could be hidden in the surgery.

(5) Finally, the most general idea is to develop a topological field theory on the base of our invariant. The specific construction of the present paper, where algebraic relations are extracted from three–dimensional Euclidean geometry, is by now means unique. We hope to present vast generalizations in further papers.

Acknowledgements. The authors J.D. and I.G.K. acknowledge support from the *Swiss National Science Foundation.* In particular, it made possible the visit of I.G.K to Geneva, where the work on this paper began.

The work of J.D. is supported by the European Community with Marie Curie Intra–European Fellowship (MEIF–CT–2006–025316). While writing the paper, J.D. visited the CRM. He thanks the CRM for hospitality.

The work of I.G.K. and E.V.M. was supported by *Russian Foundation for Basic Research*, Grant no. 04-01-96010. The work of E.V.M. was also supported by a Grant for *Young Researchers from the Government of Chelyabinsk Region*, Russia.

References

[Ati88] M. F. Atiyah, *Topological quantum field theory*, Publications Mathématiques de l'IHÉS **68** (1988), 175–186.

[BHKK01] H. U. Boden, C. M. Herald, P. A. Kirk, and E. P. Klassen, *Gauge theoretic invariants of dehn surgeries on knots*, Geom. Topol. **5** (2001), 143–226.

[Dub05] J. Dubois, *Non abelian reidemeister torsion and volume form on the* $\mathfrak{su}(2)$*-representation space of knot groups*, Ann. Institut Fourier **55** (2005), 1685–1734.

[KM02] I. G. Korepanov and E. V. Martyushev, *Distinguishing three–dimensional lens spaces* $l(7,1)$ *and* $l(7,2)$ *by means of classical pentagon equation*, J. Nonlinear Math. Phys. **9** (2002), 86–98.

[Kor01] I. G. Korepanov, *Invariants of pl-manifolds from metrized simplicial complexes. three–dimensional case*, J. Nonlin. Math. Phys. **8** (2001), 196–210.

[Kor02] ———, *Euclidean 4-simplices and invariants of four-dimensional manifolds: Ii. an algebraic complex and moves* $2 \leftrightarrow 4$, Theor. Math. Phys. **133** (2002), 1338–1347.

[Kor03] ———, *Euclidean 4-simplices and invariants of four-dimensional manifolds: Iii. moves* $1 \leftrightarrow 5$ *and related structures*, Theor. Math. Phys. **135** (2003), 601–613.

[Lic99] W. B. R. Lickorish, *Simplicial moves on complexes and manifolds*, Geom. Topol. Monographs **2** (1999), 299–320.

[Mar] E. V. Martyushev, *Ph.d. thesis*, in preparation.

[Mar04] ———, *Euclidean geometric invariants of links in 3-sphere*, Proc. Chelyabinsk Sci. Center **4** (2004), 1–5.

[Rei35] K. Reidemeister, *Homotopieringe und linsenräume*, Hamburger Abhandl. **11** (1935), 102–109.

[Tur01] V. Turaev, *Introduction to combinatorial torsions*, Birkhäuser, 2001.

Section de Mathématiques, Université de Genève CP 64, 2–4 Rue du Lièvre, CH–1211 Genève 4, Switzerland; Centre de Recerca Matematica, Apartat 50, E-08193 Bellaterra, Spain and South Ural State University, 76 Lenin Ave., 454080 Chelyabinsk, Russia

E-mail address: JDubois@crm.cat and kig@susu.ac.ru and mev@susu.ac.ru